

Assignment 10

This homework is due *Thursday* Nov 14.

There are total 19 points in this assignment. 17 points is considered 100%. If you go over 17 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge*. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 5.2, 5.3 in Bartle–Sherbert.

- (1) [3pt] (5.2.6) Let f, g be defined on \mathbb{R} and let $c \in \mathbb{R}$. suppose that $\lim_{x \rightarrow c} f = b$ and that g is continuous at b . Show that $\lim_{x \rightarrow c} g(f(x)) = g(b)$.
(Hint: (Re)define f to be b at c , apply composition of continuous functions.)
 NOTE. This statement says that \lim and a *continuous* function can be swapped: $\lim_{x \rightarrow c} g(f(x)) = g(\lim_{x \rightarrow c} f(x))$.
- (2) [2pt] (5.2.7) Give an example of a function $f : [0, 1] \rightarrow \mathbb{R}$ that is discontinuous at every point of $[0, 1]$ but such that $|f|$ is continuous on $[0, 1]$.
- (3) [3pt] (5.2.15) Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be continuous at a point c , and let $h(x) = \max\{f(x), g(x)\}$ for $x \in \mathbb{R}$. Show that $h(x) = \frac{1}{2}(f(x) + g(x) + |f(x) - g(x)|)$ for all $x \in \mathbb{R}$. Use this to show that h is continuous at c .
- (4) [2pt] (5.3.1) Let $I = [a, b]$ and let $f : I \rightarrow \mathbb{R}$ be a continuous function such that $f(x) > 0$ for all $x \in I$. Prove that there is a number $\alpha > 0$ such that $f(x) \geq \alpha$ for all $x \in I$.
- (5) [2pt] (Part of 5.3.5) Show that the polynomial $x^4 + 7x^3 - 9$ has at least two real roots.
- (6) [3pt] (5.3.6) Let f be continuous on the interval $[0, 1]$ to \mathbb{R} and such that $f(0) = f(1)$. Prove that there exists a point $c \in [0, \frac{1}{2}]$ such that $f(c) = f(c + \frac{1}{2})$. (*Hint: Consider $g(x) = f(x) - f(x + \frac{1}{2})$.*)
 NOTE. Therefore, there are, at any time, antipodal points on the earth's equator that have the same temperature.
- (7) [4pt] (5.3.13) Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous on \mathbb{R} and that $\lim_{x \rightarrow -\infty} f = \lim_{x \rightarrow \infty} f = 0$. Prove that f is bounded on \mathbb{R} and attains either a maximum or a minimum on \mathbb{R} . Give an example to show that both a maximum and a minimum need not be attained.